FRACTALS

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A report by -
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Introduction

Fractals represents a branch in mathematics which tends to abolish the barrier between mathematics and art. Fractals come under geometry and have a vast array of practical applications in the modern world.
The word fractals comes from the latin word *fractus* which means broken and is used to describe objects that are too irregular to fit into a traditional geometric setting. So, a fractal, in simple terms, may be defined as a broken geometrical shape that can be split into parts that are a scaled down version of the original shape that is such a shape which when zoomed upon closely provides us the same figure. Fractal properties include properties such as independence of scale, self similarity, complexity, and infinite detail. Fractal structures do not have a single length scale, while fractal processes cannot be characterized. the necessary and sufficient conditions for an object (or process) to possess fractal properties have not been formally defined. Indeed, fractal geometry has been described as "a collection of examples, linked by a common point of view, not an organized theory. In this paper we cover 2 of the mentioned properties.

Self Similarity

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An Object is considered self similar if a part of the object, when scaled by a factor of $k>0$ is equivalent to the object itself. For example, if we look at a broccoli, each florette of the broccoli when zoomed looks like the broccoli itself. **Same is with fractals, we can scale them up or magnify them many times and after every step we will encounter the same shape with which we started.**
Let us consider a square and we highlight a portion of the same square which is a scaled down version of the original by ½, when we magnify the same by 2 times we obtain the original square.
All this scaling down and magnifying does appear quite simple, however we do need to watch out for certain different shapes. This time let us look at a circle, now the closer we magnify the circle the flatter its curve tends to get, hence a circle is not a self similar shape. This same concept can be applied to any differentiable curve, i.e., no differentiable curve is self similar(except a line). A self similar set is one which would be invariant under more than one similarity transformations.

Let us look at the other property and also try to derive its formula for further use

A dimension is defined as the no. of coordinate points needed to specify a point within a space. The dimension of a self similar object is called as similarity dimension.
Before we set to establish the mathematical formula behind dimension of a figure let us take a look at 3 shapes of which the dimensions are quite commonly known. A line(1), a square(2), a cube(3). Let us break the line into 4 equal parts, where each part is 1/4th the original, now if we suppose the length of the line to be one, so if we were to obtain the original line we take the no. of broken lines (N=4) and multiply it by the amount with which we scaled it down(¼) to the power 1, ie, 4*(¼)^1 would give us 1. The one which is in the higher power is defined as the dimension of an object. Mathematically the formula can be defined as **$N*(1/r)^d=1$.**

Now, $N*r^d=1$, applying logarithmic function and solving it we obtain the formula, **d=Log(N)/Log(1/r)** (where d is the similarity dimension, r represents the amount with which we scaled it down and N represents the number of subdivisions of the original)

Applying the above formula to that of a square, 4 subdivisions and scaled down twice, then d=log(4)/log(2)=2. Similarly for a cube, with 27 subdivisions where each subdivision is ⅓ the original. d=log(27)/log(3)=3 and so on. Now, one of the properties of dimensions is that each whole dimension has a specific lebesgue measure attached to it. **Lebesgue measure** in simple terms is basically a measure on a Euclidean space, which assigns the conventional length, area and volume to suitable subsets of R^1,R^2 and R^3 respectively and does so for further subsets of R^n where n is a natural number. This is an important concept which we would take up later on.
Let us Now consider, a line segment having length 1. Now let us remove the middle ⅓ portion of the line and we are left with 2 segments each having a length of ⅓ , applying the same step to these 2 segments we get 4 segments each having length 1/9. Now if we keep on applying these steps we find out that the length of the initial segment approaches zero,
length(initial) = 1
length(step 1) = 1-⅓
length(step 2) = 1-⅓-2/9
length(step 3) = 1-⅓-2/9-4/27
length(step n) = 1-⅓-(2/9+4/27+8/81+16/243+......)
= $1-⅓*\Sigma(⅔)^k$
= 1-⅓*3 (using infinite GP)
= 0

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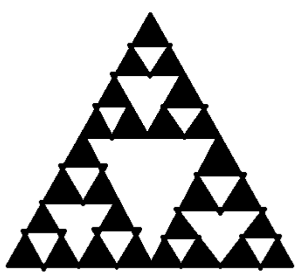
The above set is a geometric representation of a **cantor set**. A cantor set is one in which we assume a line segment having unit length and proceed to remove the middle 1/3 part in every iteration. Now an important thing to note is that although the distance between 2 consecutive segments is getting removed, the points still remain but however the length approaches zero. That means the given example is atleast a zero dimension(due to presence of points) but not a one dimension(due to absence of a measure of length, which all one dimensional bodies are supposed to have according to Lebesgue measure). So our given example lies in between zero and one dimension, this may seem absurd for a body not having a dimension in whole numbers. Before we go ahead and calculate it's dimension with the formula we got earlier let's look at a 2D example.

Consider a filled triangle ABC with PQR being the midpoints of the segments respectively. Now a new triangle PQR can be formed within ABC, so we remove this triangle from the original. Now the remaining 3 triangles within ABC also have their respective segment midpoints so we form another triangle within them and remove them, we keep on doing these so as to obtain a self similar figure which when scaled up provides us the original figure, i.e., each sub triangle when scaled up by 3 gives us ABC and each sub-sub triangle when scaled up by 3 gives us the sub triangle and so on. This is an example of a *sierpinski triangle* the figure of which is given below(ignore the immaturity in my drawing, first time doodling in paint)



Now we also mentioned that our triangle is filled, so as we keep on removing the successive sub triangles our area also keeps getting reduced.
So our area at the nth step would be(assuming initial area to be 1) -
=>$1-\frac{1}{4}-3/16-9/64-\ldots$
=>$1-1/4\Sigma(\frac{3}{4})^k$
=>0

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It is important to note that although the area keeps on decreasing indefinitely the line segments still remain, hence we know that our given example has more than one dimension(perimeter can be calculated) but due to the absence of an area parameter it's less than 2.

Now let us use the similarity dimension formula on these 2 examples.
For the sierpinski triangle, each sub triangle is formed from the midpoints of the original, that is it is scaled by ½ and we have 3 sub triangles so.
d=log(3)/log(½) = 1.585 approx.
Similarly for the cantor set,
d=log(2)/log(⅓) = 0.6309 approx

Here we note that the similarity dimensions are not whole numbers but fractions that is fractional dimensions. The idea of a fractional dimension was an important aspect in Benoit Mandelbrot's study of natural phenomena and this where he coined the term ***fractals*** in his 1977 book "Fractals : form,chance and dimension" it's from this book that we received the mathematical form of the **first defined fractal**. I highlighted this term to emphasise on the fact that fractals had been mathematically drawn before Mandelbrot gave his definition, back in the 1800s, Weirstrass gave the equation for the Weierstrass curve; a curve which is everywhere continuous and nowhere differentiable and when zoomed upon, indefinitely , keeps on giving the original curve.

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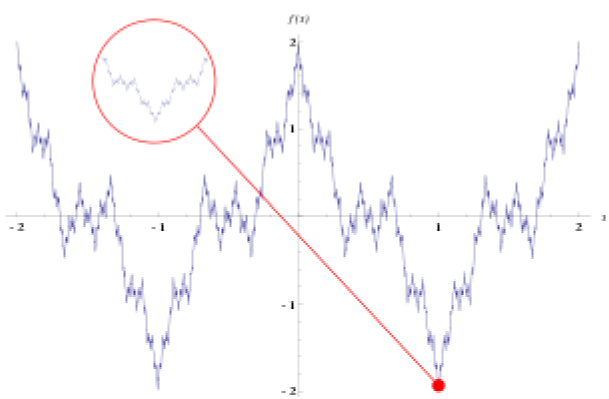
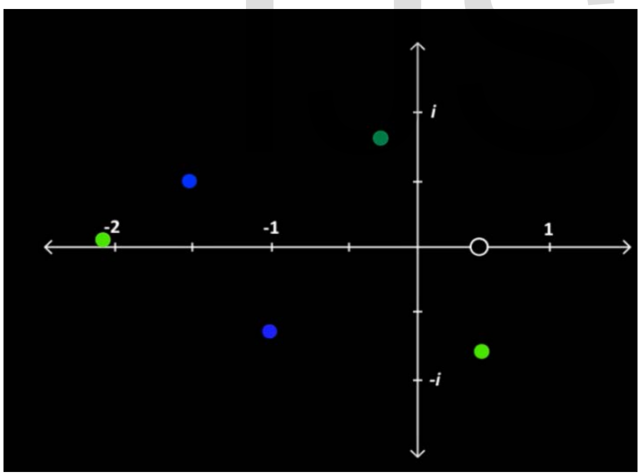


MANDELBROT AND JULIA SET

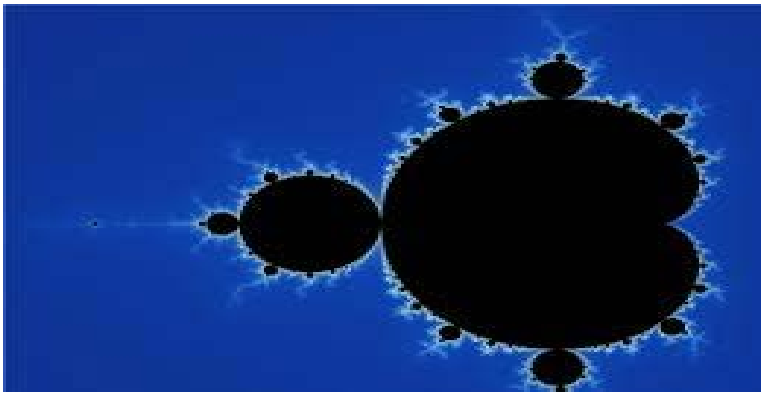
Let Z be a complex no. and $f(z_n)=z_{n-1}^2+c$ be a single quadratic function in the real plane where c is a number on the complex plane. Now the Mandelbrot and Julia sets, both are based on the idea of taking Z_0 and c and repeatedly finding the value of Z_n.

Mandelbrot Set

In this, we take $Z_0 = 0$ and allow the value of c to vary. That is we plot our graph taking the value of c as different for each iteration, in one iteration the value of c remains fixed for that particular case. We combine all the cases and use a method of coloring as given below to plot our curve.
If the Value of Zn is bounded under 2 then plot it black.
Now different values of c would give us different cases and each case would grow at a different rate , so its coloured according to the degree with which it grows



After plotting all these points we form a graph which looks like this



Zooming in on it indefinitely, would give us an infinite amount of depth in pattern which changes as much as we zoom and doesn't repeat the patterns.

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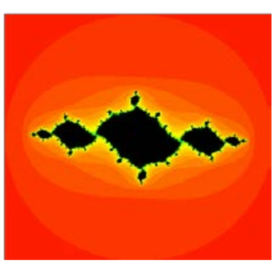
Julia Set

In this set we set the value of c for one case but allow the values of Z_0 to vary, which is not necessarily zero, and plot according to the same procedure as we adopted in the Mandelbrot set. The basic difference in the 2 is that -

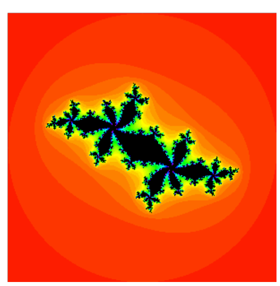
1) The Julia set shows us a repeating looped pattern whereas the mandelbrot set brings out new patterns every time we zoom further

2)There are infinite amount of Julia sets but only one Mandelbrot set(This statement is obvious as in the Mandelbrot set we take all possible values of c under Z_n and plot them together)

Different examples of Julia sets depend on the different values of C we choose to take.



c= -1.037+0.17i



c=-0.52+0.57i

(images from math.bu.edu)

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One more thing I would like to add which I discovered while conducting my research is that, as I mentioned earlier, there are infinite julia sets. Now if we choose the value of c according to a point on the mandelbrot set, zoom in on it and make a julia set with the same point the 2 graphs appear to be similar(the zoomed part in mandelbrot and the entire julia set)

Applications of Fractals

The application of this topic in the modern world is immense. It ranges from the technical fields of computer graphic imagery to the world of biological sciences and so many more. Just to give an example of its practicality, earlier biologists used traditional objects and series, under Euclidean representations, to model different aspects of nature. Such as sine waves for heart rates, cell membranes as curves or simple surfaces, conifer trees as cones etc. These figures were useful but were limited to the amount of specification. As I mentioned earlier, fractals have the property of being able to hold infinite amounts of data.

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Now since Biological structures have been found to portray multiple levels of substructures with the same general pattern repeating indefinetly, Fractals looked like the right way of representation. As mentioned by Kenkel and Walker in 1993, Fractal proved to be a unifying theme in biology as it allowed for us to generalize the fundamental concepts of dimension and length measurement .
Even in the research of evolution, scientists have witnessed emerging subpatterns in the structure of DNA and are trying to study it with the help of fractals.

Even in movies like star wars, you may notice the death star having a repeating pattern. That pattern was achieved by creating an algorithm to repeat the circular structures and showcase it on screen. At the climax of the movie "interstellar", the tesseract is made up of a pattern of "submerged rectangles" which helped in giving the illusion of infinite space.

Many scientists agree that fractal geometry is a powerful tool for uncovering the secrets of the universe. The list of known fractals systems is quite long and still continues to grow .

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